

Universal Coding for Lossless and Lossy Complementary Delivery Problems

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Abstract

This paper deals with a coding problem called complementary delivery, where messages from two correlated sources are jointly encoded and each decoder reproduces one of two messages using the other message as the side information. Both lossless and lossy universal complementary delivery coding schemes are investigated. In the lossless case, it is demonstrated that a universal complementary delivery code can be constructed by only combining two Slepian-Wolf codes. Especially, it is shown that a universal lossless complementary delivery code, for which error probability is exponentially tight, can be constructed from two linear Slepian-Wolf codes. In the lossy case, a universal complementary delivery coding scheme based on Wyner-Ziv codes is proposed. While the proposed scheme cannot attain the optimal rate-distortion trade-off in general, the rate-loss is upper bounded by a universal constant under some mild conditions. The proposed schemes allows us to apply any Slepian-Wolf and Wyner-Ziv codes to complementary delivery coding.

Index Terms

complementary delivery, multiterminal source coding, network coding, universal coding, Slepian-Wolf coding, Wyner-Ziv coding.

I. INTRODUCTION

The source coding problem for correlated information sources was initiated by Slepian and Wolf [1]. They treated the case where two information sources are encoded separately and then reproduced at the single destination. Subsequently, various coding problems derived from Slepian-Wolf coding have been considered (e.g. [2]–[5]). Corresponding lossy coding problem was studied by Wyner and Ziv [6], where they investigated the lossy coding problem when the decoder can fully observe the side information. While the messages are encoded separately in

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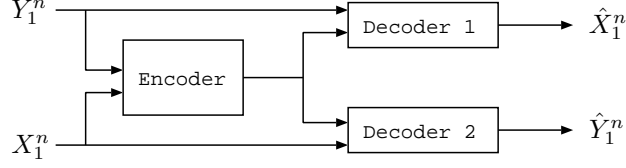


Fig. 1. Complementary delivery problem

Slepian-Wolf and Wyner-Ziv coding problems, the coding problems involving joint encoding processes has been also explored (e.g. [7]–[10]).

This paper deals with a specific coding problem involving joint encoding, which is called *complementary delivery coding*. The block diagram of the complementary delivery coding is depicted in Fig. 1. The encoder observes messages emitted from two correlated sources, and delivers these messages to two destinations (decoder 1 and 2). Each decoder reproduces one of two messages using the other message as the side information. Both lossless and lossy configurations have been considered. The lossless complementary delivery coding can be regarded as a special case of the coding problem investigated by Csiszár and Körner [11] and Wyner, Wolf and Willems [10]. Kimura *et al.* [12], [13] proposed a universal coding scheme for lossless complementary delivery based on graph coloring. The lossy complementary delivery problem was investigated by Kimura and Uyematsu [14], [15].

In this paper, we propose universal coding schemes for lossless and lossy complementary delivery coding problems. At first, we propose a simple construction of the lossless complementary delivery code based on Slepian-Wolf codes. The key idea of our coding scheme is as follows. We prepare two Slepian-Wolf codes. One of two codes is a Slepian-Wolf code for the source X with side information Y , which is used as the code from the encoder to the decoder 1. The other code is a Slepian-Wolf code for the source Y with side information X , which is used as the code from the encoder to the decoder 2. Each source is encoded separately by the corresponding Slepian-Wolf code, and then, the encoder sends the summation of two codewords. Notice that, by using the side information Y , the decoder 1 can calculate the codeword from the encoder to decoder 2. Therefore, from the summation of two codewords, decoder 1 can extract the codeword to reproduce X . The decoder 2 can reproduce Y analogously. The above mentioned scheme allows us to apply any Slepian-Wolf code to lossless complementary delivery coding. This drastically enriches the variety of complementary delivery coding. For example, we can use universal Slepian-Wolf codes (e.g. [16]–[18]). We can also apply Slepian-Wolf codes based on the low-density parity-check matrices (e.g. [19], [20]). In this paper, we demonstrate that a universal lossless complementary delivery code, for which the error probability is exponentially tight in some rate region, can be constructed by combining linear Slepian-Wolf codes [16].

Next, we propose a universal lossy complementary delivery coding scheme based on Wyner-Ziv codes [6]. Our scheme is universal in the sense that it does not depend on the joint probability distribution of the correlated sources. While our coding scheme cannot attain the optimal rate-distortion trade-off in general, the rate-loss is upper bounded by a universal constant under some mild conditions. Moreover, our scheme allows us to construct a universal lossy

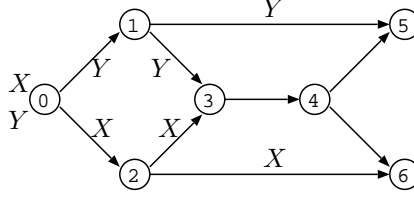


Fig. 2. The source node 0 observes the correlated sources (X, Y) , and sends the message to the sink nodes 5 and 6 over the network. The node 3 (resp. 5, 6) corresponds to the encoder (resp. decoder 1,2) in Fig. 1.

complementary delivery code by using (non-universal) Wyner-Ziv codes (e.g. [21]–[23]).

The complementary delivery coding can be regarded as a special case of the network coding [24], [25]. Let us consider the network depicted in Fig. 2. The source node 0 observes the messages emitted from the correlated sources (X, Y) , and sends the message to the sink nodes 5 and 6 over the network. Assume that the all edges except the edge between the nodes 3 and 4 have sufficiently large capacity, and thus, the output from X (resp. Y) can be delivered to the nodes 3 and 6 (resp. 3 and 5). The problem is to find the minimum capacity between the nodes 3 and 4 satisfying that the codeword needed to reproduce X and Y can be delivered to the nodes 5 and 6. Then, this problem can be regarded as the complementary delivery problem depicted in Fig. 1. The node 3 (resp. 5, 6) corresponds to the encoder (resp. decoder 1,2). The coding problem of correlated sources over a network was studied by Han [26]. In the recent years, considerable attentions have been devoted to Slepian-Wolf coding over a network (e.g. [27]–[31]). This paper shows that, for the specific network depicted in Fig. 2, the optimal code can be constructed by only combining two Slepian-Wolf codes. Further, the lossy complementary delivery investigated in this paper can be seen as a special case of lossy coding of correlated sources over a network, which is not so studied well as the lossless case.

This paper is organized as follows. In Section II, we introduce definitions and notations used in this paper. In Section III, lossless complementary delivery coding is considered. We propose a simple construction of the lossless complementary delivery code based on Slepian-Wolf codes. Further, we propose another simple coding scheme which can work in a specific case. In Section IV, lossy complementary delivery coding is considered. We propose a universal lossy complementary delivery coding scheme based on Wyner-Ziv codes. In Section V, we present our conclusions and some directions for further work.

II. PRELIMINARY

We denote by \mathbb{N} a set of positive integers $\{1, 2, \dots\}$. For a finite set S , $|S|$ denotes the cardinality of S . Throughout this paper, we take all log and exp to the base 2. We denote random variables by upper case letters such as X . Their sample values (resp. alphabets) are denoted by the corresponding lower case letters such as x (resp. calligraph letters such as \mathcal{X}). For a random variable X , P_X denotes the probability distribution of X . Similarly, for a pair of random variables (X, Y) , the joint distribution is denoted by P_{XY} and the conditional distribution of Y give X is written by $P_{Y|X}$. For each $n \in \mathbb{N}$, X^n denotes a random n -vector (X_1, X_2, \dots, X_n) , and $x^n = (x_1, \dots, x_n)$.

denotes a specific sample value in \mathcal{X}^n which is the n -th Cartesian product of \mathcal{X} . A substring of x^n is written as $x_i^j = (x_i, x_{i+1}, \dots, x_j)$ for $i \leq j$. When the dimension is clear from the context, vectors will be denoted by boldface letters such as $\mathbf{x} \in \mathcal{X}^n$.

A discrete memoryless source (DMS) is a sequence $\mathbf{X} \triangleq \{X_i\}_{i=1}^\infty$ of independent and identically distributed (i.i.d.) copies of a random variable X . For simplicity, we call a DMS $\mathbf{X} = \{X_i\}_{i=1}^\infty$ as a source X . In this paper, information theoretic quantities will be denoted following the usual conventions of the information theory literature (see, e.g. [32], [33]). The *entropy rate* of a source X is denoted by $H(X)$. For a pair (X, Y) of correlated sources X and Y , the *conditional entropy* of Y given X is denoted by $H(Y|X)$, and the mutual information between X and Y is denoted by $I(X; Y)$. The *relative entropy* or *divergence* between two probability distributions P and Q is denoted by $D(P\|Q)$.

For a given pair $(\mathbf{x}, \mathbf{y}) \in (\mathcal{X} \times \mathcal{Y})^n$ of sequences, the *joint type* of (\mathbf{x}, \mathbf{y}) is defined as the empirical distribution $Q_{\mathbf{xy}}$ of (\mathbf{x}, \mathbf{y}) , that is,

$$Q_{\mathbf{xy}}(a, b) = \frac{|\{1 \leq i \leq n : x_i = a, y_i = b\}|}{n}$$

for all $(a, b) \in \mathcal{X} \times \mathcal{Y}$ [33]. Let $\mathcal{P}_n(\mathcal{X} \times \mathcal{Y})$ be the set of all joint types of sequences in $(\mathcal{X} \times \mathcal{Y})^n$. By the type counting lemma [33, Lemma 1.2.2], we have

$$|\mathcal{P}_n(\mathcal{X} \times \mathcal{Y})| \leq (n+1)^{|\mathcal{X}||\mathcal{Y}|}. \quad (1)$$

Hence, we can define an injection $\iota_n: \mathcal{P}_n \rightarrow \{1, 2, \dots, (n+1)^{|\mathcal{X}||\mathcal{Y}|}\}$. $\iota_n(P_{\hat{X}\hat{Y}})$ is called the index assigned to $P_{\hat{X}\hat{Y}} \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Y})$.

III. LOSSLESS COMPLEMENTARY DELIVERY

A. Previous results

In this subsection, we formulate the lossless complementary delivery problem and show a fundamental bound of the coding rate.

Definition 1: A *lossless complementary delivery code* of block length n is defined by a triple of mappings $(f_n, \phi_n^{(1)}, \phi_n^{(2)})$ where

$$f_n: \mathcal{X}^n \times \mathcal{Y}^n \rightarrow \mathcal{M}_n,$$

$$\phi_n^{(1)}: \mathcal{M}_n \times \mathcal{Y}^n \rightarrow \mathcal{X}^n,$$

$$\phi_n^{(2)}: \mathcal{M}_n \times \mathcal{X}^n \rightarrow \mathcal{Y}^n,$$

where $\mathcal{M}_n = \{1, 2, \dots, \|f_n\|\}$ and $\|f_n\| < \infty$.

Definition 2: For a given pair (X, Y) of correlated sources X and Y , a rate R is said to be *losslessly-achievable*

if there exists a sequence $\{(f_n, \phi_n^{(1)}, \phi_n^{(2)})\}_{n=1}^\infty$ of codes satisfying

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \|f_n\| &\leq R, \\ \limsup_{n \rightarrow \infty} \Pr \left\{ X^n \neq \phi_n^{(1)}(f_n(X^n, Y^n), Y^n) \right\} &= 0, \\ \limsup_{n \rightarrow \infty} \Pr \left\{ Y^n \neq \phi_n^{(2)}(f_n(X^n, Y^n), X^n) \right\} &= 0. \end{aligned}$$

As a special case of results of [11] and [10], it can be shown that the infimum of the losslessly-achievable rate is given by $\max\{H(X|Y), H(Y|X)\}$. Kimura *et al.* [12] proposed the universal coding scheme based on graph coloring which can achieve any rate $R > \max\{H(X|Y), H(Y|X)\}$.

Theorem 1 (Lossless coding theorem; direct part [12]): For a given rate R , there exists a sequence $\{(f_n, \phi_n^{(1)}, \phi_n^{(2)})\}_{n=1}^\infty$ of a code such that for any (X, Y) ,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \|f_n\| \leq R$$

and

$$\begin{aligned} &\Pr \left\{ X^n \neq \phi_n^{(1)}(f_n(X^n, Y^n), Y^n) \right\} \\ &+ \Pr \left\{ Y^n \neq \phi_n^{(2)}(f_n(X^n, Y^n), X^n) \right\} \\ &\leq \exp \left\{ -n \left[\min_{P_{\hat{X}\hat{Y}} \in \overline{\mathcal{S}}_n(R)} D(P_{\hat{X}\hat{Y}} \| P_{XY}) - \zeta_n \right] \right\} \end{aligned}$$

where

$$\begin{aligned} \overline{\mathcal{S}}_n(R) &\triangleq \left\{ P_{\hat{X}\hat{Y}} \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Y}) : \right. \\ &\quad \left. \max\{H(\hat{X}|\hat{Y}), H(\hat{Y}|\hat{X})\} > R \right\} \end{aligned}$$

and

$$\begin{aligned} \zeta_n &\triangleq \frac{1}{n} \{|\mathcal{X} \times \mathcal{Y}| \log(n+1) + 1\} \\ &\rightarrow 0 \quad (n \rightarrow \infty). \end{aligned}$$

On the other hand, the next theorem shows that the error exponent of the code appeared in Theorem 1 is tight.

Theorem 2 (Lossless coding theorem; converse part [12]): For any code $(f_n, \phi_n^{(1)}, \phi_n^{(2)})$ satisfying $(1/n) \log \|f_n\| = R$, we have

$$\begin{aligned} &\Pr \left\{ X^n \neq \phi_n^{(1)}(f_n(X^n, Y^n), Y^n) \right\} \\ &+ \Pr \left\{ Y^n \neq \phi_n^{(2)}(f_n(X^n, Y^n), X^n) \right\} \\ &\geq \exp \left\{ -n \left[\min_{P_{\hat{X}\hat{Y}} \in \overline{\mathcal{S}}_n(R+\zeta_n)} D(P_{\hat{X}\hat{Y}} \| P_{XY}) + \zeta_n \right] \right\}. \end{aligned} \tag{2}$$

B. Universal coding based on Slepian-Wolf codes

As shown in Section III-A, the coding scheme proposed in [12] is universal and optimal. However, it requires the exponentially large coding table. In this subsection, we propose a simple coding scheme based on Slepian-Wolf codes.

At first, we consider Slepian-Wolf coding problem of a source X with side information Y . A Slepian-Wolf code of block length n for a source X with side information Y is defined by a pair of mappings $(g_n^{(1)}, \psi_n^{(1)})$ where

$$\begin{aligned} g_n^{(1)}: \mathcal{X}^n &\rightarrow \bar{\mathcal{M}}_n, \\ \psi_n^{(1)}: \bar{\mathcal{M}}_n \times \mathcal{Y}^n &\rightarrow \mathcal{X}^n, \end{aligned}$$

and $\bar{\mathcal{M}}_n = \{1, 2, \dots, \|g_n^{(1)}\|\}$. Similarly, we can define a Slepian-Wolf code $(g_n^{(2)}, \psi_n^{(2)})$ for a source Y with side information X . The next lemma gives a simple construction of a lossless complementary delivery code from Slepian-Wolf codes.

Lemma 1: For given Slepian-Wolf codes $(g_n^{(1)}, \psi_n^{(1)})$ and $(g_n^{(2)}, \psi_n^{(2)})$, there exists a lossless complementary delivery code $(f_n, \phi_n^{(1)}, \phi_n^{(2)})$ such that

$$\|f_n\| \leq \max\{\|g_n^{(1)}\|, \|g_n^{(2)}\|\}$$

and

$$\begin{aligned} &\Pr \left\{ X^n \neq \phi_n^{(1)}(f_n(X^n, Y^n), Y^n) \right\} \\ &\leq \Pr \left\{ X^n \neq \psi_n^{(1)}(g_n^{(1)}(X^n), Y^n) \right\}, \\ &\Pr \left\{ Y^n \neq \phi_n^{(2)}(f_n(X^n, Y^n), X^n) \right\} \\ &\leq \Pr \left\{ Y^n \neq \psi_n^{(2)}(g_n^{(2)}(Y^n), X^n) \right\}. \end{aligned}$$

Proof: Let $M_n = \max\{\|g_n^{(1)}\|, \|g_n^{(2)}\|\}$ and define f_n , $\phi_n^{(1)}$, and $\phi_n^{(2)}$ by

$$\begin{aligned} f_n(\mathbf{x}, \mathbf{y}) &\triangleq g_n^{(1)}(\mathbf{x}) \oplus g_n^{(2)}(\mathbf{y}), \\ \phi_n^{(1)}(m, \mathbf{y}) &\triangleq \psi_n^{(1)}(m \ominus g_n^{(2)}(\mathbf{y}), \mathbf{y}), \\ \phi_n^{(2)}(m, \mathbf{x}) &\triangleq \psi_n^{(2)}(m \ominus g_n^{(1)}(\mathbf{x}), \mathbf{x}), \end{aligned}$$

where \oplus (resp. \ominus) denotes the addition (resp. subtraction) in modulo M_n arithmetic. The lemma follows from the construction of the code $(f_n, \phi_n^{(1)}, \phi_n^{(2)})$. \square

Lemma 1 allows us to apply any Slepian-Wolf code to lossless complementary delivery problem. This drastically enriches the variety of complementary delivery coding. In the remaining part of this subsection, we demonstrate that a universal code which achieves the optimal rate $\max\{H(X|Y), H(Y|X)\}$ can be constructed by applying universal linear Slepian-Wolf codes [16] to Lemma 1.

Theorem 3: Assume that $\mathcal{X} = \mathcal{Y}$ and \mathcal{X} is a Galois field. Fix k ($k \leq n$) and let $R = (k/n) \log |\mathcal{X}|$. There exists a sequence $\{(f_n, \phi_n^{(1)}, \phi_n^{(2)})\}_{n=1}^{\infty}$ of lossless complementary delivery codes such that for any (X, Y) ,

$$\frac{1}{n} \log \|f_n\| = R$$

and

$$\begin{aligned} & \Pr \left\{ X^n \neq \phi_n^{(1)}(f_n(X^n, Y^n), Y^n) \right\} \\ & \leq \exp \left\{ -n (e_r^1(R, P_{XY}) - \varepsilon_n) \right\} \\ & \Pr \left\{ Y^n \neq \phi_n^{(2)}(f_n(X^n, Y^n), X^n) \right\} \\ & \leq \exp \left\{ -n (e_r^2(R, P_{XY}) - \varepsilon_n) \right\} \end{aligned}$$

where

$$\varepsilon_n \triangleq \frac{2 \log(n+1)}{n} |\mathcal{X}|^2 |\mathcal{Y}|^2$$

and

$$\begin{aligned} & e_r^1(R, P_{XY}) \\ & \triangleq \min_{P_{\tilde{X}\tilde{Y}}} \left\{ D(P_{\tilde{X}\tilde{Y}} \| P_{XY}) + \left| R - H(\tilde{X}|\tilde{Y}) \right|^+ \right\} \\ & e_r^2(R, P_{XY}) \\ & \triangleq \min_{P_{\tilde{X}\tilde{Y}}} \left\{ D(P_{\tilde{X}\tilde{Y}} \| P_{XY}) + \left| R - H(\tilde{Y}|\tilde{X}) \right|^+ \right\} \end{aligned}$$

where the minimization is over all dummy random variables \tilde{X} and \tilde{Y} with joint distribution $P_{\tilde{X}\tilde{Y}}$, and $|t|^+ \triangleq \max(t, 0)$.

Proof: Csiszár [16] showed that there exists a linear Slepian-Wolf code $(g_n^{(1)}, \psi_n^{(1)})$ for a source X with side information Y such that $g_n^{(1)}: \mathcal{X}^n \rightarrow \mathcal{X}^k$ and for any (X, Y) ,

$$\begin{aligned} & \Pr \left\{ X^n \neq \psi_n^{(1)}(g_n^{(1)}(X^n), Y^n) \right\} \\ & \leq \exp \left\{ -n (e_r^1(R, P_{XY}) - \varepsilon_n) \right\}. \end{aligned}$$

Similarly, there exists a linear Slepian-Wolf code $(g_n^{(2)}, \psi_n^{(2)})$ for a source Y with side information X such that $g_n^{(2)}: \mathcal{Y}^n \rightarrow \mathcal{Y}^k$ and for any (X, Y) ,

$$\begin{aligned} & \Pr \left\{ Y^n \neq \psi_n^{(2)}(g_n^{(2)}(Y^n), X^n) \right\} \\ & \leq \exp \left\{ -n (e_r^2(R, P_{XY}) - \varepsilon_n) \right\}. \end{aligned}$$

By applying two codes $(g_n^{(1)}, \psi_n^{(1)})$ and $(g_n^{(2)}, \psi_n^{(2)})$ to Lemma 1, we have the theorem. \square

Remark 1: As mentioned in Section I, Theorem 3 can be seen as a result of Slepian-Wolf coding over a specific network (see Fig. 2). In [31, Theorem 6], Ho *et al.* also applied the linear coding approach in [16] to Slepian-Wolf coding over a network. However, there are some differences between our results and the result of [31]. While Ho

et al. considered more general networks than the network depicted in Fig. 2, Theorem 6 of [31] dealt with the special case where there exists only one receiver. Hence, our result, Theorem 3, cannot be derived only by applying Theorem 6 of [31] to the network depicted in Fig. 2. Moreover, Lemma 1 allows us to apply not only liner but also various Slepian-Wolf codes to network coding. Further, the technique used in the proof of Lemma 1 can be applied to the lossy case which was not concerned in [31] (see Remark 4).

Remark 2: In [12], the coding scheme based on graph coloring is applied to variable-rate coding for lossless complementary delivery problem. In the same way as the approach in [12], our coding scheme can be also modified and applied to variable-rate coding. For a given pair of sequences (\mathbf{x}, \mathbf{y}) to be encoded, let \hat{X} and \hat{Y} be random variables such that $P_{\hat{X}\hat{Y}}$ is identical to the joint type of (\mathbf{x}, \mathbf{y}) . Let $R = \max\{H(\hat{X}|\hat{Y}), H(\hat{Y}|\hat{X})\}$ and $(f_n, \phi_n^{(1)}, \phi_n^{(2)})$ be a fixed-rate lossless complementary delivery code such that $(1/n) \log \|f_n\| = R$. If $\mathbf{x} = \phi_n^{(1)}(f_n(\mathbf{x}, \mathbf{y}), \mathbf{y})$ and $\mathbf{y} = \phi_n^{(2)}(f_n(\mathbf{x}, \mathbf{y}), \mathbf{x})$, then, the encoder sends the codeword consisting of the flag bit “0”, the index $\iota_n(P_{\hat{X}\hat{Y}})$ of $P_{\hat{X}\hat{Y}}$, and $f_n(\mathbf{x}, \mathbf{y})$. This codeword can be represented by using at most $1 + |\mathcal{P}_n(\mathcal{X} \times \mathcal{Y})| + R$ bits. On the other hand, if $\mathbf{x} \neq \phi_n^{(1)}(f_n(\mathbf{x}, \mathbf{y}), \mathbf{y})$ or $\mathbf{y} \neq \phi_n^{(2)}(f_n(\mathbf{x}, \mathbf{y}), \mathbf{x})$, then, the encoder sends the codeword consisting of the flag bit “1” and (\mathbf{x}, \mathbf{y}) , which can be represented by using $1 + \lceil n \log |\mathcal{X} \times \mathcal{Y}| \rceil$ bits. The overflow probability of the coding rate of this scheme can be bounded in the same way as an error probability of fixed-rate coding (see [12] for more details). Hence, it can be shown that, by using Slepian-Wolf codes, we can construct a universal variable-rate lossless complementary delivery code for which the coding rate is smaller than or equal to $\max\{H(X|Y), H(Y|X)\}$ asymptotically almost surely.

Now, we investigate the tightness of the error exponent of the proposed scheme. It is known that

$$\begin{aligned} e_r^1(R, P_{XY}) &= \min_{H(\hat{X}|\hat{Y}) \geq R} D(P_{\hat{X}\hat{Y}} \| P_{XY}) \\ e_r^2(R, P_{XY}) &= \min_{H(\hat{Y}|\hat{X}) \geq R} D(P_{\hat{X}\hat{Y}} \| P_{XY}) \end{aligned}$$

if $R \leq R_{cr}^i$ ($i = 1, 2$), where $R_{cr}^i = R_{cr}^i(P_{XY})$ is the largest R for which the curve $e_r^i(R, P_{XY})$ meets its supporting line of slope one [16]. Hence, as a corollary of Theorem 3, we have

$$\begin{aligned} &\Pr \left\{ X^n \neq \phi_n^{(1)}(f_n(X^n, Y^n), Y^n) \right\} \\ &+ \Pr \left\{ Y^n \neq \phi_n^{(2)}(f_n(X^n, Y^n), X^n) \right\} \\ &\leq 2 \exp \left\{ -n \min_{P_{\hat{X}\hat{Y}}} D(P_{\hat{X}\hat{Y}} \| P_{XY}) - \varepsilon_n \right\} \end{aligned} \quad (3)$$

for R such that $\max\{H(X|Y), H(Y|X)\} \leq R \leq \min_{i=1,2} R_{cr}^i$. By comparing (3) with (2), it can be seen that the error bound (3) is exponentially tight for R such that $\max\{H(X|Y), H(Y|X)\} \leq R \leq \min_{i=1,2} R_{cr}^i$. On the other hand, the error exponent bound for large rates can be improved in the same way as improving the error exponent of Slepian-Wolf coding.

Theorem 4: Assume that $\mathcal{X} = \mathcal{Y}$ and \mathcal{X} is a Galois field. Fix k ($k \leq n$) and let $R = (k/n) \log |\mathcal{X}|$. There exists

a sequence $\{(f_n, \phi_n^{(1)}, \phi_n^{(2)})\}_{n=1}^\infty$ of lossless complementary delivery codes such that for any (X, Y) ,

$$\frac{1}{n} \log \|f_n\| = R$$

and

$$\begin{aligned} & \Pr \left\{ X^n \neq \phi_n^{(1)}(f_n(X^n, Y^n), Y^n) \right\} \\ & \leq \exp \left\{ -n (e_x^1(R, P_{XY}) - \varepsilon_n) \right\} \\ & \Pr \left\{ Y^n \neq \phi_n^{(2)}(f_n(X^n, Y^n), X^n) \right\} \\ & \leq \exp \left\{ -n (e_x^2(R, P_{XY}) - \varepsilon_n) \right\} \end{aligned}$$

where

$$\begin{aligned} & e_x^1(R, P_{XY}) \\ & \triangleq \min_{\tilde{X}: H(\tilde{X}) \geq R} \left\{ E_{\tilde{X}} \left[-\log \sum_{x,y} \sqrt{P_{XY}(x,y) P_{XY}(x \ominus \tilde{X}, y)} \right] + R - H(\tilde{X}) \right\} \\ & e_x^2(R, P_{XY}) \\ & \triangleq \min_{\tilde{Y}: H(\tilde{Y}) \geq R} \left\{ E_{\tilde{Y}} \left[-\log \sum_{x,y} \sqrt{P_{XY}(x,y) P_{XY}(x, y \ominus \tilde{Y})} \right] + R - H(\tilde{Y}) \right\} \end{aligned}$$

where \ominus denotes the subtraction in the field $\mathcal{X} (= \mathcal{Y})$ and $E_{\tilde{X}}$ (resp. $E_{\tilde{Y}}$) denotes the expectation with respect to $P_{\tilde{X}}$ (resp. $P_{\tilde{Y}}$).

Proof: By using Slepian-Wolf codes which attain the expurgated bound [16, Theorem 3], we can prove the theorem in the same way as Theorem 3. \square

C. Binary symmetric case

In this subsection, we propose another simple coding scheme for lossless complementary delivery problem which can work in a specific case. Let $\mathcal{X} = \mathcal{Y} = \{0, 1\}$, and consider a binary symmetric source with parameter p ($0 \leq p \leq 1/2$), that is,

$$P_{XY}(xy) = \begin{cases} \frac{1-p}{2}, & \text{if } x = y, \\ \frac{p}{2}, & \text{if } x \neq y. \end{cases}$$

In this case, a simple universal lossless code gives an optimal lossless complementary delivery scheme. For a given $\mathbf{x} \in \mathcal{X}^n$ and $\mathbf{y} \in \mathcal{Y}^n$, let $\mathbf{w} \triangleq \mathbf{x} \oplus \mathbf{y}$, where \oplus denotes the addition in modulo 2 arithmetic. Then, \mathbf{w} can be regarded as an output from the source $W \triangleq X \oplus Y$, which satisfies that $P_W(0) = 1 - p$ and $P_W(1) = p$. It is well known that (see e.g. [33]) there exists a universal lossless code $(\bar{f}_n, \bar{\phi}_n)$ with rate R such that

$$\lim_{n \rightarrow \infty} \Pr \left\{ W^n \neq \bar{\phi}_n(\bar{f}_n(W^n)) \right\} = 0$$

provided that $R \geq h(p)$, where h is the binary entropy function defined as $h(t) \triangleq -t \log t - (1-t) \log(1-t)$. By using $(\bar{f}_n, \bar{\phi}_n)$, we can define the code $(f_n, \phi_n^{(1)}, \phi_n^{(2)})$ as

$$\begin{aligned} f_n(\mathbf{x}, \mathbf{y}) &\triangleq \bar{f}_n(\mathbf{x} \oplus \mathbf{y}), \\ \phi_n^{(1)}(m, \mathbf{y}) &\triangleq \bar{\phi}_n(m) \oplus \mathbf{y}, \\ \phi_n^{(2)}(m, \mathbf{x}) &\triangleq \bar{\phi}_n(m) \oplus \mathbf{x}. \end{aligned}$$

By the construction of the code, this simple code $(f_n, \phi_n^{(1)}, \phi_n^{(2)})$ is universal. Further, it achieves the optimal rate $h(p) = \max\{H(X|Y), H(Y|X)\}$ since $h(p) = H(X|Y) = H(Y|X)$. Especially, if $(\bar{f}_n, \bar{\phi}_n)$ is a lossless code for which the error exponent is tight (e.g. a code appeared in [33]), then, the error exponent of the code $(f_n, \phi_n^{(1)}, \phi_n^{(2)})$ based on $(\bar{f}_n, \bar{\phi}_n)$ is also tight, that is, $(f_n, \phi_n^{(1)}, \phi_n^{(2)})$ attains the error exponent appeared in (2). Furthermore, in the same way, we can also apply lossy codes to construct a lossy complementary delivery code (See the proof of Theorem 8 in Appendix B).

IV. LOSSY COMPLEMENTARY DELIVERY

A. Previous results

In this subsection, we formulate the lossy complementary delivery problem and show a fundamental bound of the coding rate. Let $\hat{\mathcal{X}}$ and $\hat{\mathcal{Y}}$ be reconstruction alphabets, and $d^{(1)}: \mathcal{X} \times \hat{\mathcal{X}} \rightarrow [0, d_{\max}^{(1)}]$ and $d^{(2)}: \mathcal{Y} \times \hat{\mathcal{Y}} \rightarrow [0, d_{\max}^{(2)}]$ be single-letter distortion functions ($d_{\max}^{(i)} < \infty$ ($i = 1, 2$)). Then, for each $n \in \mathbb{N}$, the normalized distortion $d_n^{(1)}(\mathbf{x}, \hat{\mathbf{x}})$ between $\mathbf{x} \in \mathcal{X}^n$ and $\hat{\mathbf{x}} \in \hat{\mathcal{X}}^n$ is defined as

$$d_n^{(1)}(\mathbf{x}, \hat{\mathbf{x}}) = \frac{1}{n} \sum_{i=1}^n d^{(1)}(x_i, \hat{x}_i).$$

For $\mathbf{y} \in \mathcal{Y}^n$ and $\hat{\mathbf{y}} \in \hat{\mathcal{Y}}^n$, $d_n^{(2)}(\mathbf{y}, \hat{\mathbf{y}})$ is defined similarly. Now, we define codes for lossy complementary delivery problem.

Definition 3: A lossy complementary delivery code of block length n is defined by a triple of mappings $(f_n, \phi_n^{(1)}, \phi_n^{(2)})$ where

$$\begin{aligned} f_n: \mathcal{X}^n \times \mathcal{Y}^n &\rightarrow \mathcal{M}_n, \\ \phi_n^{(1)}: \mathcal{M}_n \times \mathcal{Y}^n &\rightarrow \hat{\mathcal{X}}^n, \\ \phi_n^{(2)}: \mathcal{M}_n \times \mathcal{X}^n &\rightarrow \hat{\mathcal{Y}}^n, \end{aligned}$$

where $\mathcal{M}_n = \{1, 2, \dots, \|f_n\|\}$ and $\|f_n\| < \infty$.

Next, we define the achievability of rate and the optimal rate attained by lossy complementary delivery coding.

Definition 4: For a given (X, Y) and a distortion pair $(\Delta^{(1)}, \Delta^{(2)})$, a rate R is said to be $(\Delta^{(1)}, \Delta^{(2)})$ -achievable

if there exists a sequence $\{(f_n, \phi_n^{(1)}, \phi_n^{(2)})\}_{n=1}^\infty$ of codes satisfying

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \|f_n\| &\leq R, \\ \limsup_{n \rightarrow \infty} E_{XY} \left[d_n^{(1)} \left(X^n, \phi_n^{(1)}(f_n(X^n, Y^n), Y^n) \right) \right] &\leq \Delta^{(1)}, \\ \limsup_{n \rightarrow \infty} E_{XY} \left[d_n^{(2)} \left(Y^n, \phi_n^{(2)}(f_n(X^n, Y^n), X^n) \right) \right] &\leq \Delta^{(2)}, \end{aligned}$$

where E_{XY} denotes the expectation with respect to P_{XY} .

Definition 5: For a pair of sources (X, Y) and a pair of distortions $(\Delta^{(1)}, \Delta^{(2)})$, let

$$R^*(X, Y | \Delta^{(1)}, \Delta^{(2)}) \triangleq \inf \left\{ R : R \text{ is } (\Delta^{(1)}, \Delta^{(2)})\text{-achievable} \right\}.$$

Kimura and Uyematsu [14], [15] revealed the optimal achievable rate for the lossy complementary delivery.

Theorem 5 (Lossy coding theorem [14], [15]): For a given (X, Y) ,

$$\begin{aligned} R^*(X, Y | \Delta^{(1)}, \Delta^{(2)}) \\ = \min_{P_{U|XY}} [\max\{I(X; U|Y), I(Y; U|X)\}] \end{aligned}$$

where the minimization is over all the auxiliary random variable U satisfying the following properties:

- 1) $P_{XYU}(x, y, u) = P_{XY}(x, y)P_{U|XY}(u|x, y)$,
- 2) U takes a value over an alphabet \mathcal{U} satisfying $|\mathcal{U}| \leq |\mathcal{X} \times \mathcal{Y}| + 2$, and
- 3) there are functions $\varphi^{(1)}: \mathcal{U} \times \mathcal{Y} \rightarrow \hat{\mathcal{X}}$ and $\varphi^{(2)}: \mathcal{U} \times \mathcal{X} \rightarrow \hat{\mathcal{Y}}$ satisfying

$$\begin{aligned} \Delta^{(1)} &\geq E_{XYU} \left[d^{(1)}(X, \varphi^{(1)}(U, Y)) \right], \\ \Delta^{(2)} &\geq E_{XYU} \left[d^{(2)}(Y, \varphi^{(2)}(U, X)) \right]. \end{aligned}$$

B. Universal coding based on Wyner-Ziv codes

The coding scheme appeared in the direct part of the proof of Theorem 5 depends on the joint distribution P_{XY} of (X, Y) . We propose a lossy complementary delivery coding scheme which does not depend on the joint distribution.

At first, we consider Wyner-Ziv coding problem of a source X with side information Y under the distortion constraint $\Delta^{(1)}$ associated with the distortion measure $d^{(1)}: \mathcal{X} \times \hat{\mathcal{X}} \rightarrow [0, d_{\max}^{(1)}]$. A Wyner-Ziv code of block length n for a source X with side information Y is defined by a pair of mappings $(g_n^{(1)}, \psi_n^{(1)})$ where

$$\begin{aligned} g_n^{(1)}: \mathcal{X}^n &\rightarrow \bar{\mathcal{M}}_n, \\ \psi_n^{(1)}: \bar{\mathcal{M}}_n \times \mathcal{Y}^n &\rightarrow \hat{\mathcal{X}}^n, \end{aligned}$$

and $\bar{\mathcal{M}}_n = \{1, 2, \dots, \|g_n^{(1)}\|\}$. Define $R_{WZ}(X, Y | d^{(1)}, \Delta^{(1)})$ by

$$R_{WZ}(X, Y | d^{(1)}, \Delta^{(1)}) \triangleq \min_{P_{U|X}} \{I(X; U) - I(Y; U)\}$$

where the minimization is over all the random variables U satisfying the following properties:

- 1) $P_{XYU}(x, y, u) = P_{U|X}(u, x)P_{XY}(x, y)$,
- 2) $|\mathcal{U}| \leq |\mathcal{X}| + 1$, and
- 3) there exists a function $\varphi: \mathcal{U} \times \mathcal{Y} \rightarrow \hat{\mathcal{X}}$ satisfying that

$$E_{XYU}[d^{(1)}(X, \varphi(U, Y))] \leq \Delta^{(1)}. \quad (4)$$

To simplify the notation, we denote $R_{WZ}(X, Y|d^{(1)}, \Delta^{(1)})$ by $R_{WZ}^{(1)}(\Delta^{(1)}, P_{XY})$. It is known that the optimal coding rate which can be achieved by a Wyner-Ziv code for a source X with side information Y under the distortion constraint $\Delta^{(1)}$ is given by $R_{WZ}^{(1)}(\Delta^{(1)}, P_{XY})$.

Theorem 6 ([6], [33]): For any $\delta > 0$ and any (X, Y) , there exists $l_0 = l_0(\delta, d_{\max}^{(1)}, |\mathcal{X}|, |\mathcal{Y}|)$ such that for any $l \geq l_0$ there exists a code $(g_l^{(1)}, \psi_l^{(1)})$ satisfying

$$\frac{1}{l} \log \|g_l^{(1)}\| \leq R_{WZ}^{(1)}(\Delta^{(1)}, P_{XY}) + \delta$$

and

$$\Pr \left\{ d_l^{(1)} \left(X^l, \psi_l^{(1)} \left(g_l^{(1)}(X^l), Y^l \right) \right) > \Delta^{(1)} \right\} \leq \delta.$$

Remark 3: While $(g_l^{(1)}, \psi_l^{(1)})$ depends on (X, Y) , the virtue of the method of types [33] allows us to choose the block size l which depends only on $\delta, d_{\max}^{(1)}, |\mathcal{X}|$, and $|\mathcal{Y}|$.

In a similar manner to the above discussion, we can consider Wyner-Ziv coding problem of a source Y with side information X under the distortion constraint $\Delta^{(2)}$ associated with the distortion measure $d^{(2)}: \mathcal{Y} \times \hat{\mathcal{Y}} \rightarrow [0, d_{\max}^{(2)}]$. We denote $R_{WZ}(Y, X|d^{(2)}, \Delta^{(2)})$ by $R_{WZ}^{(2)}(\Delta^{(2)}, P_{XY})$.

Now, we describe a universal lossy complementary delivery coding scheme based on Wyner-Ziv codes. Fix $\gamma > 0$ and $R > 0$. Choose $\delta > 0$ such that $\delta < \gamma/4$ and $4\delta d_{\max}^{(i)} < \gamma$ ($i = 1, 2$). By Theorem 6, we can choose $l = l(\delta, d_{\max}^{(1)}, d_{\max}^{(2)}, |\mathcal{X}|, |\mathcal{Y}|)$ sufficiently large so that, for any correlated sources (\hat{X}, \hat{Y}) , there are $(g_l^{(1)}, \psi_l^{(1)})$ and $(g_l^{(2)}, \psi_l^{(2)})$ satisfying

$$\frac{1}{l} \log \|g_l^{(i)}\| \leq R_{WZ}^{(i)}(\Delta^{(i)}, P_{\hat{X}\hat{Y}}) + \delta, \quad i = 1, 2 \quad (5)$$

and

$$\Pr \{ \hat{X}^l \hat{Y}^l \notin \Gamma_l(P_{\hat{X}\hat{Y}}) \} \leq 2\delta \quad (6)$$

where

$$\Gamma_l(P_{\hat{X}\hat{Y}}) \triangleq \left\{ (x^l, y^l) : \begin{aligned} & d_l^{(1)} \left(x^l, \psi_l^{(1)}(g_l^{(1)}(x^l), y^l) \right) \leq \Delta^{(1)}, \\ & d_l^{(2)} \left(y^l, \psi_l^{(2)}(g_l^{(2)}(y^l), x^l) \right) \leq \Delta^{(2)} \end{aligned} \right\}.$$

Note that $(g_l^{(i)}, \psi_l^{(i)})$ ($i = 1, 2$) may depend on $P_{\hat{X}\hat{Y}}$. Especially, for each joint type $P_{\hat{X}\hat{Y}}$, we can choose the pair of codes $\{(g_l^{(i)}, \psi_l^{(i)})\}_{i=1,2}$ satisfying (5) and (6). For each $n \in \mathbb{N}$, fix a correspondence between $\mathcal{P}_n(\mathcal{X} \times \mathcal{Y})$ and the set of pairs of Wyner-Ziv codes so that the pair $\{(g_l^{(i)}, \psi_l^{(i)})\}_{i=1,2}$ corresponding to $P_{\hat{X}\hat{Y}} \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Y})$ satisfies (5) and (6)¹. Let $\bar{M}_l \triangleq 2^{l(R+3\gamma/4)}$. Let n be so large that $n > l$ and $(|\mathcal{X}||\mathcal{Y}|/n) \log(n+1) < \gamma/4$. In the followings, we assume that $n = Tl$ ($T \in \mathbb{N}$) for simplicity.

At first, we describe the encoding scheme. For a given (x^n, y^n) , find $P_{\hat{X}\hat{Y}} \in \mathcal{P}_n$ such that²

$$\max_{i=1,2} R_{WZ}^{(i)}(\Delta^{(i)}, P_{\hat{X}\hat{Y}}) \leq R + \gamma/2 \quad (7)$$

and

$$\left| \left\{ t : (x_{tl+1}^{(t+1)l}, y_{tl+1}^{(t+1)l}) \notin \Gamma_l(P_{\hat{X}\hat{Y}}) \right\} \right| \leq 4\delta T. \quad (8)$$

Note that $P_{\hat{X}\hat{Y}}$ is not necessarily the joint type of (x^n, y^n) . If there is no $P_{\hat{X}\hat{Y}} \in \mathcal{P}_n$ satisfying (7) and (8), then error is declared. If there exists $P_{\hat{X}\hat{Y}} \in \mathcal{P}_n$ satisfying (7) and (8), then find the pair of Wyner-Ziv codes corresponding to $P_{\hat{X}\hat{Y}}$, that is, $\{(g_l^{(i)}, \psi_l^{(i)})\}_{i=1,2}$ satisfying (5) and (6). Parse (x^n, y^n) into T blocks of size l , and then, encode each block as

$$m_t = g_l^{(1)}(x_{tl+1}^{(t+1)l}) \oplus g_l^{(2)}(y_{tl+1}^{(t+1)l}), \quad t = 0, \dots, T-1$$

where \oplus denotes the addition in modulo \bar{M}_l arithmetic (Note that $\bar{M}_l \geq \max_{i=1,2} \|g_l^{(i)}\|$). Then, the codeword assigned to (x^n, y^n) is $(\iota(P_{\hat{X}\hat{Y}}), m_0, \dots, m_{T-1})$. Since (1) holds, the codeword can be described by using $n(R+\gamma)$ bits because

$$\log(n+1)^{|\mathcal{X}||\mathcal{Y}|} + T \log \bar{M}_l \leq n(R+\gamma).$$

Next, we describe the decoding scheme. We only describe the decoder $\phi_n^{(1)}$ which outputs the reproduction sequence $\hat{x}^n \in \hat{\mathcal{X}}^n$ by using the codeword $(\iota(P_{\hat{X}\hat{Y}}), m_0, \dots, m_{T-1})$ and the side information y^n . The decoder $\phi_n^{(2)}$ can be defined analogously. $\phi_n^{(1)}$ decodes the index $\iota(P_{\hat{X}\hat{Y}})$ at first, and then, computes \hat{x}^n as

$$\hat{x}_{tl+1}^{(t+1)l} \triangleq \psi_n^{(1)} \left(m_t \ominus g_l^{(2)}(y_{tl+1}^{(t+1)l}), y_{tl+1}^{(t+1)l} \right), \quad t = 0, \dots, T-1$$

where $\psi_n^{(1)}$ and $g_l^{(2)}$ are the mappings corresponding to $P_{\hat{X}\hat{Y}}$, and \ominus denotes the subtraction in modulo \bar{M}_l arithmetic.

The next theorem shows the performance of the coding scheme described above.

Theorem 7: Fix $\gamma > 0$ and $R > 0$. There exists a sequence $\{(f_n, \phi_n^{(1)}, \phi_n^{(2)})\}_{n=1}^\infty$ of lossy complementary delivery codes which satisfies the following property: If (X, Y) satisfies

$$R \geq \max_{i=1,2} R_{WZ}^{(i)}(\Delta^{(i)}, P_{XY})$$

then, for sufficiently large n ,

$$\frac{1}{n} \log \|f_n\| \leq R + \gamma$$

¹If there are two or more pairs of codes satisfying (5) and (6) for a joint type $P_{\hat{X}\hat{Y}}$, then choose one of them arbitrarily and assign it to $P_{\hat{X}\hat{Y}}$

²If there are two or more joint types satisfying the conditions, choose one of them arbitrarily.

and

$$\begin{aligned} \Pr \left\{ d_n^{(1)} \left(X^n, \phi_n^{(1)} (f_n(X^n, Y^n), Y^n) \right) > \Delta^{(1)} + \gamma \right\} &\leq \gamma, \\ \Pr \left\{ d_n^{(2)} \left(Y^n, \phi_n^{(2)} (f_n(X^n, Y^n), X^n) \right) > \Delta^{(2)} + \gamma \right\} &\leq \gamma. \end{aligned}$$

Remark 4: The proposed scheme is universal in the sense that the scheme does not depend on the probability distribution P_{XY} of (X, Y) . To deal with some technical difficulties in evaluating the performance of the code, we adopt the coding scheme which parses the sequence into blocks of fixed length l and then encodes each block. On the other hand, if we know the the joint distribution P_{XY} , we can avoid technical difficulties. In fact, a (non-universal) lossy complementary delivery code can be constructed by combining two Wyner-Ziv codes $(g_n^{(1)}, \psi_n^{(1)})$ and $(g_n^{(2)}, \psi_n^{(2)})$ in the same way as a lossless complementary delivery code is constructed by Slepian-Wolf codes (Lemma 1). In the lossless case, we can use universal Slepian-Wolf codes to construct a universal lossless complementary delivery code. However, as long as the authors know, no universal Wyner-Ziv code has been proposed (While universal Wyner-Ziv coding was recently studied in [23], [34], it is assumed that the conditional distribution $P_{Y|X}$ of the source is known).

The proof of the theorem will be given in Appendix A.

Our coding scheme allows us to construct a universal lossy complementary delivery coding scheme based on (non-universal) Wyner-Ziv codes. Especially, we can apply practical Wyner-Ziv codes (e.g. [21]–[23]) to universal lossy complementary delivery. However, our scheme cannot attain the optimal rate appeared in Theorem 5 in general.

Theorem 8: There exists (X, Y) such that

$$\max_{i=1,2} R_{WZ}^{(i)}(\Delta^{(i)}, P_{XY}) > R^*(X, Y | \Delta^{(1)}, \Delta^{(2)}).$$

Remark 5: In the proof of Theorem 8, we give an example where a simple coding scheme appeared in Section III-C can attain the optimal rate $R^*(X, Y | \Delta^{(1)}, \Delta^{(2)})$. See Appendix B for more details.

On the other hand, we can show that the loss of the coding rate can be bounded by a universal constant under some conditions. Let $\mathcal{X} = \hat{\mathcal{X}} = \{1, 2, \dots, M^{(1)}\}$ and $\mathcal{Y} = \hat{\mathcal{Y}} = \{1, 2, \dots, M^{(2)}\}$. Suppose that $d^{(i)}$ ($i = 1, 2$) is a *balanced* distortion measure [35], that is,

$$d^{(i)}(a, \hat{a}) = \bar{d}^{(i)}(a \ominus \hat{a}), \quad \forall a, \hat{a} \in \{1, \dots, M^{(i)}\}$$

for some $\bar{d}^{(i)} : \{1, \dots, M^{(i)}\} \rightarrow [0, d_{\max}^{(i)}]$, where \ominus denotes modulo- $M^{(i)}$ subtraction. Let $C^{(i)}$ ($i = 1, 2$) be the *minimax capacity* [36], defined as

$$C^{(i)}(\Delta^{(i)}) = \inf_{N: E_N[\bar{d}^{(i)}(N)] \leq \Delta^{(i)}} \sup_{\substack{W: W \perp N, \\ E_W[\bar{d}^{(i)}(W)] \leq \Delta^{(i)}}} I(W; W \oplus N)$$

where N and W are random variables on $\{1, 2, \dots, M^{(i)}\}$, \perp denotes statistical independence, and \oplus denotes modulo- $M^{(i)}$ addition. The next theorem gives the bound on the rate-loss of our scheme.

Theorem 9: Suppose that $\mathcal{X} = \hat{\mathcal{X}}$ and $\mathcal{Y} = \hat{\mathcal{Y}}$. If $d^{(1)}$ and $d^{(2)}$ are balanced distortion measures, then

$$\max_{i=1,2} R_{WZ}^{(i)}(\Delta^{(i)}, P_{XY}) \leq R^*(X, Y | \Delta^{(1)}, \Delta^{(2)}) + C^{(i^*)}(\Delta^{(i^*)})$$

where $i^* = \arg \max_{i=1,2} R_{WZ}^{(i)}(\Delta^{(i)}, P_{XY})$.

Proofs of Theorem 8 and Theorem 9 will be given in Appendix B.

V. CONCLUSION

We proposed a universal lossless (resp. lossy) complementary delivery coding scheme based on Slepian-Wolf (resp. Wyner-Ziv) codes. It was demonstrated that a universal lossless complementary delivery code, for which error probability is exponentially tight, can be constructed by only combining two linear Slepian-Wolf codes. On the other hand, proposed lossy complementary delivery coding scheme cannot attain the optimal rate generally, while it does not depend on the distribution of the source. The rate-loss of our lossy coding scheme was evaluated. We also propose another simple coding scheme which can work for a binary symmetric source.

Further work includes extensions to generalized complementary delivery network [13]. Another important work is to construct a universal lossy complementary delivery code which attains the optimal rate.

APPENDIX

PROOFS OF THEOREMS

A. Proof of Theorem 7

Before proving Theorem 7, we introduce some lemmas. In this appendix, the variational distance between P_{XY} and $P_{\hat{X}\hat{Y}}$ is denoted by $\rho(P_{XY}, P_{\hat{X}\hat{Y}})$.

Lemma 2: For any $\delta > 0$ and $l \in \mathbb{N}$, there exists $\epsilon_1 = \epsilon_1(l, \delta, |\mathcal{X}|, |\mathcal{Y}|) > 0$ such that if

$$\rho(P_{XY}, P_{\hat{X}\hat{Y}}) \leq \epsilon_1$$

then,

$$\rho(P_{X^l Y^l}, P_{\hat{X}^l \hat{Y}^l}) \leq \delta.$$

Proof: If $\rho(P_{XY}, P_{\hat{X}\hat{Y}}) \leq \epsilon_1$, then for any $(x^l, y^l) \in (\mathcal{X} \times \mathcal{Y})^l$,

$$\begin{aligned} P_{X^l Y^l}(x^l, y^l) &= \prod_{i=1}^l P_{XY}(x_i, y_i) \\ &\leq \prod_{i=1}^l \{P_{\hat{X}\hat{Y}}(x_i, y_i) + \epsilon_1\} \\ &\leq \prod_{i=1}^l P_{\hat{X}\hat{Y}}(x_i, y_i) + \delta(\epsilon_1, l) \\ &= P_{\hat{X}^l \hat{Y}^l}(x^l, y^l) + \delta(\epsilon_1, l) \end{aligned}$$

where $\delta(\epsilon_1, l) \rightarrow 0$ as $\epsilon_1 \rightarrow 0$. Similarly, we have $P_{X^l Y^l}(x^l, y^l) \geq P_{\hat{X}^l \hat{Y}^l}(x^l, y^l) - \delta(\epsilon_1, l)$. Hence, we have the lemma. \square

Lemma 3: 1) For any $\gamma > 0$ and any $P_{\hat{X}\hat{Y}}$, there exists $\zeta > 0$ satisfying

$$R_{WZ}^{(i)}(\Delta^{(i)}, P_{\hat{X}\hat{Y}}) \leq R_{WZ}^{(i)}(\Delta^{(i)} + \zeta, P_{\hat{X}\hat{Y}}) + \gamma/4, \quad i = 1, 2.$$

2) For any $\gamma > 0$, $\zeta > 0$ and any (X, Y) , there exists $\epsilon_2 > 0$ such that if $\rho(P_{XY}, P_{\hat{X}\hat{Y}}) < \epsilon_2$ then

$$R_{WZ}^{(i)}(\Delta^{(i)} + \zeta, P_{\hat{X}\hat{Y}}) \leq R_{WZ}^{(i)}(\Delta^{(i)}, P_{XY}) + \gamma/4, \quad i = 1, 2.$$

Proof: The first part of the lemma follows from the fact that $R_{WZ}^{(i)}(\Delta^{(i)}, P_{\hat{X}\hat{Y}})$ is continuous in $\Delta^{(i)}$ [6].

By the definition of $R_{WZ}^{(1)}(\Delta^{(1)}, P_{XY})$, we can choose $P_{U|X}$ and φ satisfying

$$R_{WZ}^{(1)}(\Delta^{(1)}, P_{XY}) = I(X; U) - I(Y; U)$$

and (4). If $\rho(P_{XY}, P_{\hat{X}\hat{Y}})$ is sufficiently small, then $P_{\hat{X}\hat{Y}}$ satisfies that

$$I(\hat{X}; U) - I(\hat{Y}; U) \leq I(X; U) - I(Y; U) + \gamma/4$$

and

$$E_{\hat{X}\hat{Y}U}[d^{(1)}(\hat{X}, \varphi(U, \hat{Y}))] \leq \Delta^{(1)} + \zeta.$$

Hence, we have

$$R_{WZ}^{(1)}(\Delta^{(1)} + \zeta, P_{\hat{X}\hat{Y}}) \leq R_{WZ}^{(1)}(\Delta^{(1)}, P_{XY}) + \gamma/4.$$

Similarly, we can prove that

$$R_{WZ}^{(2)}(\Delta^{(2)} + \zeta, P_{\hat{X}\hat{Y}}) \leq R_{WZ}^{(2)}(\Delta^{(2)}, P_{XY}) + \gamma/4$$

provided that $\rho(P_{XY}, P_{\hat{X}\hat{Y}})$ is sufficiently small. \square

Proof of Theorem 7: We prove the theorem by showing that the code defined in Section IV-B satisfies the property appeared in the theorem.

Let $\delta > 0$ and $l \in \mathbb{N}$ be numbers satisfying the conditions appeared in the description of coding scheme. Let $n = Tl \in \mathbb{N}$ be sufficiently large.

Suppose that there exists $P_{\hat{X}\hat{Y}} \in \mathcal{P}_n$ satisfying (8). Then,

$$\begin{aligned} d_n^{(1)}(x^n, \phi_n^{(1)}(f_n(x^n, y^n), y^n)) &\leq \frac{1}{n} \left\{ 4T\delta l d_{\max}^{(1)} + Tl\Delta^{(1)} \right\} \\ &= \Delta^{(1)} + 4\delta d_{\max}^{(1)} \\ &\leq \Delta^{(1)} + \gamma. \end{aligned}$$

Similarly,

$$d_n^{(2)}(y^n, \phi_n^{(2)}(f_n(x^n, y^n), x^n)) \leq \Delta^{(2)} + \gamma.$$

Hence, to prove the theorem, it is sufficient to show that, for sufficiently large n , we can find $P_{\hat{X}\hat{Y}} \in \mathcal{P}_n$ satisfying (7) and (8) with probability greater than $1 - \gamma$.

Choose ϵ_1 (resp. ζ , ϵ_2) satisfying Lemma 2 (resp. Lemma 3). Fix $\epsilon > 0$ such that $\epsilon < \epsilon_i$ ($i = 1, 2$). If n is sufficiently large, there exists a joint type $P_{\hat{X}\hat{Y}} \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Y})$ satisfying

$$\rho(P_{XY}, P_{\hat{X}\hat{Y}}) \leq \epsilon. \quad (9)$$

By Lemma 3, we have

$$R_{WZ}^{(i)}(\Delta^{(i)}, P_{\hat{X}\hat{Y}}) \leq R_{WZ}^{(i)}(\Delta^{(i)}, P_{XY}) + \gamma/2, \quad i = 1, 2.$$

Since $R \geq \max_{i=1,2} R_{WZ}^{(i)}(\Delta^{(i)}, P_{XY})$, $P_{\hat{X}\hat{Y}}$ satisfies (7). In the followings, we prove that $P_{\hat{X}\hat{Y}}$ also satisfies (8) with probability greater than $1 - \gamma$. By Lemma 2, (6), and (9),

$$P_{X^l Y^l}(\Gamma_l^c) \leq P_{\hat{X}^l \hat{Y}^l}(\Gamma_l^c) + \delta \leq 3\delta \quad (10)$$

where Γ_l^c denotes the complement of $\Gamma_l(P_{\hat{X}\hat{Y}})$. Let B_t ($t = 0, 1, \dots, T-1$) be random variables defined by

$$B_t \triangleq \begin{cases} 1, & (X_{tl+1}^{(t+1)l}, Y_{tl+1}^{(t+1)l}) \in \Gamma_l^c, \\ 0, & \text{otherwise.} \end{cases}$$

Since (10) implies $E_{X^l Y^l}[B_i] \leq 3\delta$, by the law of large numbers, we have

$$\lim_{T \rightarrow \infty} \Pr \left\{ \frac{1}{T} \sum_{t=0}^{T-1} B_t > 3\delta + \delta \right\} = 0.$$

Hence, if $n = Tl$ is sufficiently large, then we can find $P_{\hat{X}\hat{Y}} \in \mathcal{P}_n$ satisfying (7) and (8) with probability greater than $1 - \gamma$. \square

B. Proofs of Theorem 8 and Theorem 9

Before proving Theorem 8 and Theorem 9, we introduce some notations. Let $R_{both}^{(1)}(\Delta^{(1)}, P_{XY})$ be the optimal rate of the lossy source coding problem with side information at the encoder and the decoder [6], that is,

$$R_{both}^{(1)}(\Delta^{(1)}, P_{XY}) \triangleq \min_{P_{Z^{(1)}|XY}} I(X; Z^{(1)}|Y)$$

where the minimization is with respect to all random variables $Z^{(1)}$ such that $P_{XYZ^{(1)}}(x, y, z) = P_{XY}(x, y)P_{Z^{(1)}|XY}(z|x, y)$ is the probability distribution on $\mathcal{X} \times \mathcal{Y} \times \hat{\mathcal{X}}$ satisfying $\sum_{x,y,z} d^{(1)}(x, z)P_{XYZ^{(1)}}(x, y, z) \leq \Delta^{(1)}$. Similarly, let

$$R_{both}^{(2)}(\Delta^{(2)}, P_{XY}) \triangleq \min_{P_{Z^{(2)}|XY}} I(Y; Z^{(2)}|X).$$

Note that

$$\begin{aligned} & \max_{i=1,2} R_{both}^{(i)}(\Delta^{(i)}, P_{XY}) \\ & \leq \min_{P_{U|XY}} [\max\{I(X; U|Y), I(Y; U|X)\}] \\ & = R^*(X, Y|\Delta^{(1)}, \Delta^{(2)}) \end{aligned} \quad (11)$$

where $\min_{P_{U|XY}}$ is taken over U satisfying the properties appeared in Theorem 5.

Proof of Theorem 8: Let $\mathcal{X} = \mathcal{Y} = \hat{\mathcal{X}} = \hat{\mathcal{Y}} = \{0, 1\}$, and consider a binary symmetric source with parameter p ($0 < p < 1/2$). Let $d^{(1)}$ and $d^{(2)}$ be the Hamming distortion measure, that is $d^{(i)}(x, \hat{x}) = 0$ if $x = \hat{x}$ and $d^{(i)}(x, \hat{x}) = 1$ otherwise. Let $\Delta^{(1)} = \Delta^{(2)} = \Delta$ ($\Delta < p$).

For a given $\mathbf{x} \in \mathcal{X}^n$ and $\mathbf{y} \in \mathcal{Y}^n$, let $\mathbf{w} \triangleq \mathbf{x} \oplus \mathbf{y}$, where \oplus denotes the addition in modulo 2 arithmetic. Then, \mathbf{w} can be regarded as an output from the source $W \triangleq X \oplus Y$, which satisfies that $P_W(0) = 1 - p$ and $P_W(1) = p$.

It is known that [35] there exists a lossy code $(\bar{g}_n, \bar{\psi}_n)$ with rate $h(p) - h(\Delta)$ satisfying that

$$\lim_{n \rightarrow \infty} \Pr \left\{ d_n^{(1)}(W^n, \bar{\psi}_n(\bar{g}_n(W^n))) > \Delta \right\} = 0.$$

Based on $(\bar{g}_n, \bar{\psi}_n)$, define the code $(f_n, \phi_n^{(1)}, \phi_n^{(2)})$ as

$$\begin{aligned} f_n(\mathbf{x}, \mathbf{y}) &\triangleq \bar{g}_n(\mathbf{x} \oplus \mathbf{y}), \\ \phi_n^{(1)}(m, \mathbf{y}) &\triangleq \bar{\psi}_n(m) \oplus \mathbf{y}, \\ \phi_n^{(2)}(m, \mathbf{x}) &\triangleq \bar{\psi}_n(m) \oplus \mathbf{x}. \end{aligned}$$

Since

$$d_n^{(1)}(\mathbf{x}, \phi_n^{(1)}(m, \mathbf{y})) = d_n^{(1)}(\mathbf{w}, \bar{\psi}_n(\bar{g}_n(\mathbf{w})))$$

and

$$d_n^{(2)}(\mathbf{y}, \phi_n^{(2)}(m, \mathbf{x})) = d_n^{(2)}(\mathbf{w}, \bar{\psi}_n(\bar{g}_n(\mathbf{w}))),$$

the code $(f_n, \phi_n^{(1)}, \phi_n^{(2)})$ satisfies the distortion constraints. This fact indicates that

$$h(p) - h(\Delta) \geq R^*(X, Y | \Delta^{(1)}, \Delta^{(2)}). \quad (12)$$

Further, by the result of lossy source coding with side information at the encoder and decoder [6], we have

$$R_{both}^{(i)}(\Delta^{(i)}, P_{XY}) = h(p) - h(\Delta), \quad i = 1, 2.$$

Since (11) holds, we have

$$h(p) - h(\Delta) \leq R^*(X, Y | \Delta^{(1)}, \Delta^{(2)}). \quad (13)$$

By combining (12) and (13), we have

$$h(p) - h(\Delta) = R^*(X, Y | \Delta^{(1)}, \Delta^{(2)}). \quad (14)$$

On the other hand, by the result of Wyner-Ziv coding problem [6], we have for each $i = 1, 2$,

$$R_{WZ}^{(i)}(\Delta^{(i)}, P_{XY}) = \inf_{\theta, \beta} \{ \theta [h(p * \beta) - h(\beta)] \} \quad (15)$$

where the infimum is with respect to all θ, β such that $0 \leq \theta \leq 1$, $0 \leq \beta < p$, and $\Delta = \theta\beta + (1 - \theta)p$. (15) and (14) indicate that $\max_{i=1,2} R_{WZ}^{(i)}(\Delta^{(i)}, P_{XY}) > R^*(X, Y | \Delta^{(1)}, \Delta^{(2)})$ for all $0 < p < 1/2$. \square

Proof of Theorem 9: Since (11) holds, we can bound the rate loss as

$$\begin{aligned}
& \max_{i=1,2} R_{WZ}^{(i)}(\Delta^{(i)}, P_{XY}) - R^*(X, Y | \Delta^{(1)}, \Delta^{(2)}) \\
& \leq R_{WZ}^{(i^*)}(\Delta^{(i^*)}, P_{XY}) - \max_{i=1,2} R_{both}^{(i)}(\Delta^{(i)}, P_{XY}) \\
& \leq R_{WZ}^{(i^*)}(\Delta^{(i^*)}, P_{XY}) - R_{both}^{(i^*)}(\Delta^{(i^*)}, P_{XY}).
\end{aligned}$$

It is known that the difference $R_{WZ}^{(i^*)}(\Delta^{(i^*)}, P_{XY}) - R_{both}^{(i^*)}(\Delta^{(i^*)}, P_{XY})$ is bounded by a universal constant $C^{(i^*)}(\Delta^{(i^*)})$ [36]. \square

REFERENCES

- [1] D. Slepian and J. K. Wolf, "Noiseless coding of correlated information sources," *IEEE Trans. Inf. Theory*, vol. IT-19, no. 4, pp. 471–480, Jul. 1973.
- [2] A. D. Wyner, "On source coding with side information at the decoder," *IEEE Trans. Inf. Theory*, vol. IT-21, no. 3, pp. 294–300, May 1975.
- [3] R. F. Ahlswede and J. Körner, "Source coding with side information and a converse for degraded broadcast channels," *IEEE Trans. Inf. Theory*, vol. IT-21, no. 6, pp. 629–637, Nov. 1975.
- [4] A. Sgarro, "Source coding with side information at several decoders," *IEEE Trans. Inf. Theory*, vol. IT-23, no. 2, pp. 179–182, Mar. 1977.
- [5] J. Körner and K. Marton, "Images of a set via two channels and their role in multi-user communication," *IEEE Trans. Inf. Theory*, vol. IT-23, no. 6, pp. 751–761, Nov. 1977.
- [6] A. D. Wyner and J. Ziv, "The rate-distortion function for source coding with side information at the decoder," *IEEE Trans. Inf. Theory*, vol. IT-22, no. 1, pp. 1–10, Jan. 1976.
- [7] R. M. Gray and A. D. Wyner, "Source coding for a simple network," *Bell Syst. Tech. J.*, vol. 53, pp. 1681–1721, Nov. 1974.
- [8] H. Yamamoto, "Source coding theory for cascade and branching communication systems," *IEEE Trans. Inf. Theory*, vol. 27, no. 3, pp. 299–308, May 1981.
- [9] —, "Source coding theory for a triangular communication system," *IEEE Trans. Inf. Theory*, vol. 42, no. 3, pp. 848–853, May 1996.
- [10] A. D. Wyner, J. K. Wolf, and F. M. J. Willems, "Communicating via a processing broadcast satellite," *IEEE Trans. Inf. Theory*, vol. 48, no. 6, pp. 1243–1249, 2002.
- [11] I. Csiszár and J. Körner, "Towards a general theory of source networks," *IEEE Trans. Inf. Theory*, vol. IT-26, no. 2, pp. 155–165, Mar. 1980.
- [12] A. Kimura, T. Uyematsu, and S. Kuzuoka, "Universal coding for correlated sources with complementary delivery," *IEICE Trans. Fundamentals*, vol. E90-A, no. 9, pp. 1840–1847, Sep. 2007.
- [13] A. Kimura, T. Uyematsu, S. Kuzuoka, and S. Watanabe, "Universal source coding over generalized complementary delivery networks," *IEEE Trans. Inf. Theory*, submitted for publication. [Online]. Available: <http://arxiv.org/abs/0710.4987>
- [14] A. Kimura and T. Uyematsu, "Multiterminal source coding with complementary delivery," in *Proc. of International Symposium on Information Theory and its Applications (ISITA2006)*, Seoul, Korea, Oct. 2006, pp. 189–194.
- [15] —, "Multiterminal source coding with complementary delivery," *IEICE Trans. Fundamentals*, submitted for publication.
- [16] I. Csiszár, "Linear codes for sources and source networks: Error exponents, universal coding," *IEEE Trans. Inf. Theory*, vol. IT-28, no. 4, pp. 585–592, Jul. 1982.
- [17] Y. Oohama and T. S. Han, "Universal coding for the Slepian-Wolf data compression system and the strong converse theorem," *IEEE Trans. Inf. Theory*, vol. 40, no. 6, pp. 1908–1919, Nov. 1994.
- [18] T. Uyematsu, "An algebraic construction of codes for Slepian-Wolf source networks," *IEEE Trans. Inf. Theory*, vol. 47, no. 7, pp. 3082–3088, Nov. 2001.
- [19] A. D. Liveris, Z. Xiong, and C. N. Georgiades, "Compression of binary sources with side information at the decoder using LDPC codes," *IEEE Commun. Lett.*, vol. 6, no. 10, pp. 440–442, Oct. 2002.

- [20] J. Muramatsu, T. Uyematsu, and T. Wadayama, "Low-density parity-check matrices for coding of correlated sources," *IEEE Trans. Inf. Theory*, vol. 51, no. 10, pp. 3645–3654, Oct. 2005.
- [21] R. Zamir, S. Shamai, and U. Erez, "Nested linear/lattice codes for structured multiterminal binning," *IEEE Trans. Inf. Theory*, vol. 48, no. 6, pp. 1250–1276, Jun. 2002.
- [22] S. S. Pradhan and K. Ramchandran, "Distributed source coding using syndromes (DISCUS): Design and construction," *IEEE Trans. Inf. Theory*, vol. 49, no. 3, pp. 626–643, Mar. 2003.
- [23] S. Jalali, S. Verdú, and T. Weissman, "A universal Wyner-Ziv scheme for discrete sources," in *Proc. of 2007 IEEE International Symposium on Information Theory (ISIT2007)*, Nice, France, Jun. 2007.
- [24] R. Ahlswede, N. Cai, S.-Y. R. Li, and R. W. Yeung, "Network information flow," *IEEE Trans. Inf. Theory*, vol. 46, no. 4, pp. 1204–1216, Jul. 2000.
- [25] S.-Y. R. Li, R. W. Yeung, and N. Cai, "Linear network coding," *IEEE Trans. Inf. Theory*, vol. 49, no. 2, pp. 371–381, Feb. 2003.
- [26] T. S. Han, "Slepian-Wolf-Cover theorem for networks of channels," *Information and Control*, vol. 47, no. 1, pp. 67–83, Oct. 1980.
- [27] R. Cristescu, B. Beferull-Lozano, and M. Vetterli, "Networked Slepian-Wolf: theory, algorithms, and scaling laws," *IEEE Trans. Inf. Theory*, vol. 51, no. 12, pp. 4057–4073, Dec. 2005.
- [28] J. Barros and S. D. Servetto, "Network information flow with correlated sources," *IEEE Trans. Inf. Theory*, vol. 52, no. 1, pp. 155–170, Jan. 2006.
- [29] Y. Wu, V. Stanković, Z. Xiong, and S.-Y. Kung, "On practical design for joint distributed source and network coding," in *Proc. 1st Workshop on Network Coding, Theory, and Applications (NetCod2005)*, Riva del Garda, Italy, Apr. 2005.
- [30] A. Ramamoorthy, K. Jain, P. A. Chou, and M. Effros, "Separating distributed source coding from network coding," *IEEE Trans. Inf. Theory*, vol. 52, no. 6, pp. 2785–2795, Jun. 2006.
- [31] T. Ho, M. Médard, R. Koetter, D. R. Karger, M. Effros, J. Shi, and B. Leong, "A random linear network coding approach to multicast," *IEEE Trans. Inf. Theory*, vol. 52, no. 10, pp. 4413–4430, Oct. 2006.
- [32] T. M. Cover and J. A. Thomas, *Elements of Information Theory*. New York: Wiley, 1991.
- [33] I. Csiszár and J. Körner, *Information Theory: Coding Theorems for Discrete Memoryless Systems*. New York: Academic, 1981.
- [34] N. Merhav and J. Ziv, "On the Wyner-Ziv problem for individual sequences," *IEEE Trans. Inf. Theory*, vol. 52, no. 3, pp. 867–873, Mar. 2006.
- [35] T. Berger, *Rate Distortion Theory: A Mathematical Basis for Data Compression*. New Jersey: Prentice-Hall, Inc., 1971.
- [36] R. Zamir, "The rate loss in the Wyner-Ziv problem," *IEEE Trans. Inf. Theory*, vol. 42, no. 6, pp. 2073–2084, Nov. 1996.